Deduction, Induction, and Statistical Inference

Kevin T. Kelly and Konstantin Genin

If the theory in question is deterministic and the experimental data are inexact but infallible, then it is possible to draw deductive inferences from data to theory: i.e., if the observations are ruled out by the theory, then the theory is deductively refuted (it must be false).

But what if the experimental data are stochastic, as all real-valued observations are, due to random causes of error? On one standard, but rigid view, there are no deductive relations between theory and stochastic data, because every random sample is logically compatible with every stochastic theory—disastrously misleading data are merely improbable. Thus, no observation can deductively verify or refute any stochastic hypothesis. Alas, all real scientific inference is stochastic, due to measurement error. So nothing theoretical is deducible from statistical phenomena. Thus, all statistical inference is inductive.

Addition is the exemplar of deduction. But a desktop calculator has some small quantum chance of returning the wrong answer—a chance that becomes practically bothersome for solid state circuits operating in the vicinity of Jupiter's radiation belt, for example. Every concrete inferential mechanism, no matter how thoroughly shielded, is subject to some such tiny chance of error. Therefore, on the narrow view, *all* actual inference is inductive. True deduction is only to be found in an idealized, Platonic domain of proofs that subsist on their own, without need to generate or check them.

We recommend a more useful way to talk: a deductive inference process is an inference process that generates its conclusions with a guaranteed, low chance of error. By that standard, the desktop calculator is, after all, a deductive inference process. Inductive inference processes are non-deductive, in the sense that they do not produce their conclusions under any non-trivial, uniform bound on chance of error. A deductive inference problem is a problem solvable by a deductive procedure. Otherwise, the problem is properly inductive. The reform is obvious. But its consequences are deep and far-reaching.

The reformed way of talking bridges the deterministic and statistical domains in a way that the idealized talk does not. For example, suppose that a fully deterministic, discrete device produces 0s and 1s forever and let H_0 be the hypothesis that the device produces only 0s for eternity. Let H_1 be the contrary hypothesis. Then the inference from observations $(0, 0, 0, 1)$ to H_1 is deductive, (no chance of error), whereas the inference from observations $(0, 0, 0, 0)$ to H_0 is inductive (e.g., unit chance of error in possible world $(0, 0, 0, 0, 1, \ldots)$. Everyone would allow in this case that H_0 is deductively refutable and that H_1 is deductively verifiable. Statistical hypothesis testing is closely analogous. Consider the "sharp" hypothesis H_0 that the mean of a normal distribution is exactly 0, and let H_1 be the contrary hypothesis. One can set up a standard t-test at significance α to refute H_0 deductively and to verify H_1 deductively. One infers H_1 iff the observed sample falls in the test's rejection zone, which is set up to have at most chance α in each world in which H_0 is true. One suspends judgment if the sample falls outside the rejection zone. If H_1 is true, then the power (chance of rejection) increases to 1 eventually as the sampling distributions satisfying H_0 "pull away" from the actual sampling distribution. But, according to the traditional, narrow way of talking, deterministic verification is deductive and statistical verification is inductive. On our proposal, H_1 is deductively verifiable iff there is a procedure that is guaranteed to infer H_1 with high chance, eventually, in each possible world in which it is true, and that infers H_1 with only a low chance at each time in each possible world in which H_1 is false. Thus, H_1 is deductively verifiable in both cases.

Deductive refutability of H_0 is just deductive verifiability of $\neg H_0$. Hypothesis H_0 is not deductively verifiable in the deterministic problem, since inferring that one will always see 0s after seeing n 0s in a row leaves one open to a unit chance of error if the first 1 occurs at stage $n+1$. That is the usual way the problem of induction arises in philosophical discussions. But something very similar happens in the statistical problem as well. Suppose that a statistical procedure were to *infer* H_0 with high chance in a probabilistic world in which H_0 is true, as any deductive verifier of H_0 must. Then there exists a world in which the mean is close to 0 and the chance of erroneously inferring H_0 is still very high. So the procedure fails to verify H_0. Thus, there is no deductive verifier for H_0 in either case—*inferring* a sharp null hypothesis is an inductive inference. The usual, textbook admonition not to believe (i.e., to infer) the null hypothesis is, therefore, an instance of inductive skepticism, just like refusal to infer "always 0".

The narrow view is right about one thing: the deep analogy between the deterministic and the statistical problem is not a matter of logic or of probability. It is *topological.* The deductively verifiable propositions constitute the open sets of a topological space in both the deterministic and in several familiar statistical settings. The deductively refutable propositions are topologically closed. We propose, as a general thesis, that topological openness and closed-ness exhaust the scope of deductive methodology.

Topology also provides a unified perspective on inductive inference. The learnable propositions (the ones for which there exists a method that converges [in probability] to the truth) are characterizable topologically. Between deduction and mere learnability is optimally direct learnability, defined in terms of "straightest possible" convergence. Empirical simplicity can be defined topologically and Ockham's razor can be shown to be necessary for straightest possible convergence to the truth. All of that, and more, follows naturally from the more liberal and practical conception of deductive methodology that we now find to be indispensable for thinking about scientific method.