How Inductive is Bayesian Conditioning?

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Abstract

Bayesian conditioning is widely considered to license inductive inferences to universal hypotheses. However, several authors [Kelly, 1996, Shear et al., 2017] have called attention to a sense in which those inferences are essentially deductive: if H has high credence after conditioning on E, then the material condition $E \supset H$ has even higher prior probability. In this note, I show that a similar feature attends Jeffrey conditioning. Furthermore, I briefly address the extent of non-deductive *undermining* of prior beliefs.

Bayesian conditioning is widely considered to license inductive inferences to universal hypotheses. Indeed, the posterior probability of universal hypotheses approaches unity as favorable instances accumulate. However, several authors have called attention to a sense in which those inferences are essentially deductive [Kelly, 1996, Shear et al., 2017]. That feature is brought out rather starkly by the *Lockean thesis*, which connects quantitative and qualitative, or "full", belief. The Lockean agent has a probabilistic credence function $P(\cdot)$ over an algebra of propositions. The Lockean *believes* a proposition H iff $P(H) \geq t$ for some $t \in (.5, 1]$, i.e. if her credence in H meets some threshold t. Although the thesis is controversial, we adopt it here uncritically because it sharply illustrates the relevant feature of conditioning.

Consider the following theorem of the probability calculus:

Lemma 1. *$P(H|E) \leq P(E \supset H)$.*

Shear et al. [2017] observe the following consequence of Lemma 1:

Lemma 2 (Shear et al. [2017], Proposition 5)**.** *Suppose that a Lockean agent updates by conditioning. If she believes H after conditioning on E, then she believed the material conditional $E \supset H$ prior to conditioning.*

Lemma 2 has the following anti-inductive import: any belief that the Lockean comes to have after conditioning, she could have arrived at by adding the evidence to her prior beliefs and closing under logical consequence. For example, suppose the Lockean comes to believe that all ravens are black after observing n black ravens. By Lemma 2, her prior beliefs included the material conditional:

if not all ravens are black, then I will observe a non-black raven among the first n. While perhaps there is nothing objectionable *per se* about such an agent, there is something deeply anti-inductivist about the thesis that all rational inferences to new beliefs can be reconstructed as deductive inferences from prior beliefs plus the conditioning evidence. Schurz [2011] objects to a similar feature of AGM belief revision:

> Inductive generalizations as well as abductive conjectures accompany belief expansions by new observations, in science as well as in common sense cognitions. After observing several instances of a 'constant conjunction', humans almost automatically form the corresponding inductive generalization; and after performing a new experimental result sufficiently many times, experimental scientists proclaim the discovery of a new empirical law ... AGM-type expansion is not at all creative but merely additive: it simply adds the new information and forms the deductive closure, but never generates new (non-logically entailed) hypotheses.

Similarly, every 'expansion' of belief by the Lockean agent can be reconstructed as a deductive inference from new evidence and old beliefs. One might think that this is a feature only of the extremal notion of conditioning, and that Jeffrey conditioning may not be subject to the same critique.[1] The following Lemma shows that this is not the case.

Lemma 3. *Suppose that a Lockean agent updates by Jeffrey conditioning. If Jeffrey conditioning raises the agents credence in the evidence E, and she believes H after conditioning, then she believed the material conditional $E \supset H$ prior to conditioning on E.*

Proof of Lemma 3. Suppose that $P'(E)$ is the new degree of credence in the evidence and that $0 < P(E) \leq P'(E)$. Proposition H is above threshold t after Jeffrey conditioning on E to degree $P'(E)$ iff

$$
\begin{aligned}
\text{iff } t &\leq P(H|E) \cdot P'(E) + P(H|\neg E) \cdot (1 - P'(E)) \\
\text{only if } t &\leq P(H|E) \cdot P'(E) + (1 - P'(E)) \\
\text{iff } P'(E) &\leq P(H|E) \cdot P'(E) + (1 - t) \\
\text{iff } 1 &\leq P(H|E) + \frac{1}{P'(E)}(1 - t) \\
\text{iff } P(E) &\leq P(H \wedge E) + \frac{P(E)}{P'(E)}(1 - t) \\
\text{only if } P(E) &\leq P(H \wedge E) + (1 - t) \\
\text{iff } P(H \wedge E) + P(E \wedge \neg H) &\leq P(H \wedge E) + (1 - t) \\
\text{iff } P(E \wedge \neg H) &\leq (1 - t) \\
\text{iff } P(E \supset H) &\geq t.
\end{aligned}
$$

[1] Branden Fitelson made this suggestion to the author, in private communication.

□

Lemma 3 demonstrates that this is a pervasive feature of Bayesian updating, understood broadly enough to include Jeffrey conditioning.

Shear et al. [2017] retort that one would be too hasty to conclude that Lockean updating is essentially deductive:

> ... if any proposition newly learned by a Lockean could have been learned by deduction using the new evidence and old beliefs, then it may seem that the inductive apparatus plays an inessential role in learning. However, we suspect that this inference is a bit too quick ... acquiring new evidence can undermine an agent's old beliefs and, thus, render them unfit for use in such an inference.

In other words, although any *new* belief could have been acquired by deduction from the evidence and old beliefs, old beliefs can be *lost* after updating in a way that cannot be deductively reconstructed. Shear et al. [2017] provide an illustrative example where just such non-deductive undermining occurs. However, the extent of non-deductive undermining must be qualified. Shear et al. [2017] also prove the following theorem, for which I give a simplified demonstration.

Lemma 4 (Shear et al. [2017] Theorem 1)**.** *Let ϕ^{-1} be the inverse of the golden ratio ($\approx .618$). If $P(E) < \phi^{-1}$ and $P(H|E) < \phi^{-1}$, then $P(E \supset \neg H) > \phi^{-1}$.*

Proof. The inverse of the golden ratio is the unique positive real such that $(\phi^{-1})^2 = 1 - \phi^{-1}$. Suppose $P(H|E) < \phi^{-1}$ and $P(E) < \phi^{-1}$. Let $P(H \wedge E) = x$ and $P(\neg H \wedge E) = y$. Then:

$$P(H|E) = \frac{x}{x+y} < \phi^{-1} \text{ and } P(E) = x + y < \phi^{-1},$$

From which it follows that:

$$x < (\phi^{-1})^2 = 1 - \phi^{-1},$$

and that $1 - x = P(E \supset \neg H) > \phi^{-1}$. □

For a Lockean agent with belief threshold in the interval $(.5, \phi^{-1}]$ who does not already believe the evidence E, Lemma 4 demonstrates the following: if she does not believe H after updating on E, then she had prior belief in the material conditional $E \supset \neg H$. Therefore, all non-deductive undermining occurs either (1) for agents that already believe the conditioning evidence, or (2) for agents with thresholds in the interval $(\phi^{-1}, 1)$.[2]

[2] It is straightforward to show that the result holds for Jeffey conditioning when $P(E)/P'(E) < \phi^{-1}$.

References

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